

# On the Energy Problem in General Relativity\*

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## Abstract

The Energy Problem (EP) in General Relativity (GR) is analyzed in the context of GR's axiomatic inconsistencies. EP is classified according to its local and global aspects. The local aspects of the EP include noncovariance of the energy-momentum pseudotensor (EMPT) of the gravitational field, non-uniqueness of the EMPT, asymmetry of EMPT and vanishing metric energy-momentum tensor. The global aspect of the EP relates to the lack of integral conservation laws due to the general difficulties in defining invariant integrals of tensors in non-Euclidean space. These difficulties are related to the lack of precise definition of a reference frame in the GR. A reference frame is defined here as a differential manifold with an affine connection. The resulting unique decomposition of the Levi-Civita connection into its affine and nonmetric parts allows for a covariant definition of the gravitational energy-momentum tensor. It is pointed out that the invariance of the Lagrangian (or action functional) is a necessary but not sufficient condition to secure the covariance of the Lagrange-Euler field theory. A rigorous definition of the Lagrange Field Structure (LFS) on differential manifolds is proposed. A covariant generalization of the first Noether theorem for LFS is obtained. Different approaches to the EP are discussed.

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1. The EP has different aspects related to the local (differential) and global (integral) conservation laws. The local aspects are:
  - (a) Noncovariance of the energy-momentum pseudotensor (EMPT) of a gravitational field both as a mathematics problem in itself and, in its physical setting, as the Lorentz-Bauer-Schrödinger paradox.
  - (b) Non-uniqueness of the EMPT — there are the EMPTs of Einstein, Lorentz-Møller, Landau-Lifshitz (e.g. [2]) and others.
  - (c) Asymmetry of most of these EMPTs as well as of the canonical EMPT; the only symmetric one [3] is not even a pseudotensor.
  - (d) The metric energy-momentum tensor (EMT) which is always real and symmetric is identically equal to zero by virtue of the field equations. (The canonical EMPT – which differs from the symmetric one by the covariant divergence of the spin tensor – is also equal to zero [2, 4].)

The global aspect is that an invariant integral of any symmetrical 2-tensor in non-Euclidian space does not exist (the noninvariant integration generally used in GR is not valid despite the fact that it gives correct results in a few special cases) i.e., the global conservation laws depend on spatial symmetry (because one can define invariant integration only in symmetrical spaces). Noether Theory, which is the most consistent approach for considering conservation characteristics of the field, cannot be applied in its classical version because of its noncovariance. Therefore, we do not have a covariant, unique and conservative EMT for a gravitational field in GR. But even if we would have one, we would still not have global (integral) invariant quantities for an asymmetrical field.

2. Bianchi's identity  $D_\nu T_\mu^{\nu} = 0$ , where  $T_\mu^{\nu}$  is the EMT of matter and  $D_\nu$  is the covariant derivative with respect to Riemannian connection  $\Gamma_{\mu\nu}^{\tau}$ , can be rewritten in the form

$$\partial_\nu \left( T_\mu^{\nu} + t_\mu^{\nu} \right) \tag{1}$$

where  $t_\mu^{\nu}$  is the EMPT of the gravitational field and  $\partial_\nu$  is a partial derivative. Because of the way we defined  $t_\mu^{\nu}$ , it is not a tensor – the procedure of picking out a partial derivative from a covariant one is

not invariant. By using a superpotential  $\Psi_\mu^{\lambda\nu}$ ,  $t_\mu^v = \partial_\tau \Psi_\mu^{\tau\nu}$ ,  $t_\mu^v$  is defined noncovariantly from the very beginning. In the same way, the canonical EMPT is not a tensor because to form it we take a partial (noncovariant) derivative of the Lagrangian.

The geometrical structure of GR is  $\langle M, g, \Gamma \rangle$ , where  $M$  is a differential manifold,  $g$  is a metric on  $M$  and  $\Gamma$  is a Riemannian connection on  $M$  such that  $D_\tau g_{\mu\nu} = 0$ . The metric  $g_{\mu\nu}$  cannot serve as a potential of any field at all; if it did we would have a metrical EMT equal to zero which is unacceptable (it means that there is no field). Nor can the connection  $\Gamma_{\mu\nu}^\tau$  be a potential of a field because it is not a covariant object and includes some extra information about the coordinate system. So, if one takes  $\Gamma_{\mu\nu}^\tau$  as a potential of the gravitational field, the field is nonlocalized by definition.

The analysis of the integral conservation laws, which are valid only in symmetrical spaces, leads to the same conclusions – if the gravitational field is described by the metric  $g_{\mu\nu}$  of Riemannian space and symmetry of the space reflects the symmetry of the field, then the conservation laws exist only for symmetrical fields.

Another serious problem in GR is the definition of reference frames. Since they are not defined in GR, everyone's understanding of them is something different. We should mention that there is no justification to think that reference frame and coordinate system are one and the same. It has been proven and illustrated in many works (e.g.[5]) that a coordinate system is strictly a formal mathematical object, which has no physical, and very little mathematical, meaning, just as a reference frame is a real physical object.

In any case, we take as an axiom (because the truth of it is obvious to us) that a reference frame is a differentiable manifold with an affine connection  $\nabla$ , which also plays the role of the inertia field potential. As we saw, the geometrical structure  $\langle M, g, \Gamma \rangle$  seems to leave no room for both the reference frame and the gravitational field.

Fortunately, it is not quite so. For each Riemannian connection and metric  $g$ , there exists an affine connection  $\nabla$  such that  $\Gamma = \nabla + S$ , where  $S$  is the nonmetric part of  $\nabla$  (called tensor of nonmetricity). So the geometrical structure of GR is actually  $\langle M, g, \nabla, S \rangle$ . As soon as we assume that  $\nabla$  is an inertia field potential, it can easily be shown

within the framework of GR that  $S$  is a gravitational field itself [6]. This solves the problem immediately. Instead of using partial derivatives, we should use covariant derivatives  $\nabla_\mu$  with respect to the affine connection  $\nabla$ . If we now define an EMT of a gravitational field as  $t_\mu^v = \nabla_\tau \Psi_\mu^{\tau v}$ , where  $\Psi_\mu^{\tau v}$  is a tensor “superpotential,” it gives us covariant differential conservation laws:

$$\nabla_v (T_\mu^v + t_\mu^v) = \frac{1}{2} \{ \nabla_v, \nabla_\tau \} \Psi_\mu^{\tau v} \quad (2)$$

Existence of integral conservation laws does not depend on the symmetry of a gravitational field, but on the symmetry of the chosen reference frame. So, in the inertial reference frame there are always ten covariant conservative quantities.

3. As has been noted previously [6], the Lagrange formalism generally used in GR is not rigorous enough. Really, for the Lagrangian  $L=L(\phi, \partial_\mu\phi, \dots)$  (where  $\phi$  is some field), which is covariant but depends on noncovariant arguments, the canonical momentum of the field  $\pi^\mu = L/\partial(\partial_\mu\phi)$  turns out to be noncovariant (for simplicity, only first partials are indicated). This immediately leads to noncovariance of the canonical EMT  $T_\mu^v = \pi^v \partial_\mu\phi - \delta_\mu^v L$ , where  $\delta_\mu^v$  is the Kronecker tensor. This situation occurs in GR, where the Hilbert Lagrangian  $L = R(g_{\mu\nu}, \partial_\tau g_{\mu\nu}, \partial_\sigma \partial_\tau g_{\mu\nu})$  ( $R$  is the Riemannian curvature scalar) results in noncovariance of the EMT of the gravitational field.

Thus, invariance of the Lagrangian, or more rigorously, invariance of the action functional is not a sufficient condition to secure the covariance of the theory, but it has mistakenly been thought of as such. This misunderstanding has come about due to the absence of a rigorous definition of an invariant Lagrange field structure on a manifold. Here we shall provide this structure. Let  $M$  be a differentiable manifold with affine connection  $\nabla$  and metric  $g$ . Then let  $\phi$  be a differentiable tensor (or spinor) field on  $M$  and  $J_k(\phi)$  be the  $k$ -th order jet bundle over  $M$ . Let us construct a morphism  $L : J_k(\phi) \rightarrow R$ .

The triplet  $\Lambda = \langle M, \phi, L \rangle$  will be called a Lagrange Field Structure (LFS) on the differentiable manifold with Lagrangian  $L$ . Let  $\xi$  be a chart with field of definition  $U$ , then  $L = L(\phi, \nabla_\mu\phi, \nabla_v\nabla_\mu\phi, \dots)$ , where  $\nabla_\mu$  is the covariant derivative with respect to affine connection  $\nabla$ . So

in GR we have to write  $L = R(g_{\mu\nu}, \nabla_\tau g_{\mu\nu}, \nabla_\sigma \nabla_\tau g_{\mu\nu})$ , ( $\nabla_\tau g_{\mu\nu} \neq 0$ ) or  $L = R(g_{\mu\nu}, S_{\mu\nu}^\tau, \nabla_\sigma S_{\mu\nu}^\tau)$ .

Now we have  $\pi = L/\partial(\partial_v \phi)$ ,  $T_\mu^v = \pi^v \partial_\mu \phi - \delta_\mu^v L$  and

$$\nabla_\mu T_\nu^\mu = \pi^\tau \{\nabla_\tau, \nabla_\nu\} \phi \quad (3)$$

As we mentioned above, the Noether theory cannot be applied to GR in its noncovariant form. Due to this, we provided covariant generalization of the first Noether theorem for GR, based on the invariant definition of LFS. As a result of it, the following theorem has been obtained:

Let  $\Lambda = \langle M, \phi, L \rangle$  be a LFS. If action functional  $S[\phi]$  is invariant under a finite  $r$ -parametrical group Gr, then the  $r$  linearly independent combinations of Lagrange derivatives become divergences up to an additive geometry-dependent term, which plays a role similar to the additional sources in a nonhomogeneous Euler-Lagrange equation.

4. Generally speaking, three points of view on EP are possible: (1) The EP is not a problem at all, but rather the specific property of GR related to the Principle of Equivalence [7]; (2) The EP is a fatal problem of the theory and GR must be replaced by another theory free of the EP (like different versions of bimetrism, e.g. [8,9]); and (3) The EP is a serious problem, rooted in some incorrectness of GR, but which should be solved within the framework of GR.

The first point of view is in error because it is based on the incorrect assumption that reference frame and coordinate system are one and the same. The second point of view would be right if we would not be able to solve the problem within the framework of GR. Since we have shown here a few possible ways to solve the problem, there is no reason to reject GR.

In fact, we did not modify GR, but resolved intrinsic contradictions based on GR principles. These corrections result in a theory which differs minimally from GR and which is not an alternative, but rather the correct form of GR.

## References

1. Lorentz, H.A. Veral. Kn. Akad. Wet., Amsterdam, 25, 1380, 1916.

2. Einstein, A. Sitz. Preuss. Akad. Wiss., 1, 154, 1918.
3. Landau, L., Lifshitz, E. Field Theory, Moscow, 1962.
4. Einstein, A. Sitz. Preuss. Akad. Wiss., 2, 11, 1916.
5. Rodichev, V.I. Theory of Gravitation in an Orthogonal Frame, Nauka, Moscow, 1974.
6. Poltorak, A. GR9 Conference Abstracts, Jena, 2, 516, 1980. (Available on [www.arXiv.org](http://www.arXiv.org) as gr-qc/0403050)
7. Misner, C., Thorne, K., Wheeler, J. Gravitation, Freeman, San Francisco, 1973, p. 466.
8. Rosen, N. Phys. Rev., 57, 147, 1940.
9. Logunov, A., Folomeshkin, V. Theor. Math. Phys., Moscow, 32, 2, 174, 1977.